Spacetime foam and high-energy photons

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Abstract

It is shown that high-energy astrophysics can provide information on the small-scale structure of spacetime.

1 Introduction

The idea that spacetime on very small distance scales is not perfectly smooth has a long history. In modern times, the idea is often referred to as having a "spacetime foam" instead of the smooth Minkowski manifold (Wheeler, 1957, 1968; Hawking, 1978; Horowitz, 1991; Visser, 1996; Dowker et al., 2004; Hu, 2005).

Several theoretical arguments for and against a foam-like structure of spacetime have been given. (Spacetime would, for example, correspond to a topologically trivial manifold, a multiply connected manifold, a causal point set, a fermionic quantum vacuum, or something else.) But, ultimately, the question remains experimental: *what precisely <u>is</u> the small-scale structure of spacetime?*

Needless to say, this fundamental question is far from being answered. However, it has been recently realized that high-energy astrophysics may give valuable bounds or perhaps even clues. See, e.g., Amelino-Camelia (2000) and Jacobson et al. (2006) for two reviews.

The present contribution illustrates this astrophysics approach by giving a brief summary of some of our own work (Klinkhamer and Rupp, 2004, 2005). Of course, this is a very subjective selection, but it may, at least, give a concrete example of some of the current research.

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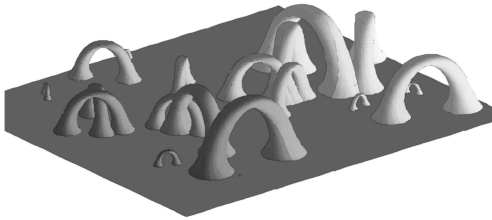cr@particle.uni-karlsruhe.de (Christian Rupp).

Fig. 1. View of a constant-time slice of a hypothetical version of spacetime foam, with one spatial dimension suppressed. Illustrated is a collection of "wormholes" (Wheeler, 1957). Points near the two "mouths" of an individual wormhole can be connected either through the flat part of space or though the wormhole "throat" (here shown as a tube rising above the plane). The lengths of the wormhole throats can be arbitrarily small (for a single wormhole, a visualization would require bending the plane).

The procedure followed is in principle straightforward and can be summarized by the following "flow chart":

START: assume a particular small-scale structure of spacetime or, at least, one characteristic (see Fig. 1 for an artist's impression);

STEP 1: calculate the effective photon model;

STEP 2: calculate the modified photon dispersion relation in the limit of large wave lengths;

STEP 3: compare with the astrophysical data and, if necessary, return to START.

In this write-up, we only sketch STEPS 1 and 2 and focus on STEP 3. The crucial points are presented in the main text and some back-of-the-envelope derivations of the experimental limits are relegated to the appendices.

2 Photon model and dispersion relation

STEP 1 mentioned in the Introduction is conceptually and technically the most difficult of the three. But this calculation is far from being completed and can as well be skipped here. We simply introduce a "random" (time-independent) background field g to mimic the anomalous effects of a multiply connected (static) spacetime foam, generalizing the result for a single wormhole (Klinkhamer and Rupp, 2004). The physics of this type of anomaly has been reviewed in Klinkhamer (2005).

The photon model is now defined in terms of the standard gauge field $A_\mu(x)$ over the *auxiliary* manifold $\mathbb{R}^4$ with Minkowski metric $(\eta_{\mu\nu}) \equiv \text{diag}(1, -1, -1, -1)$. (The real spacetime manifold M is assumed to look like Fig. 1, but here we are only interested in long-distance effects and approximate M by $\mathbb{R}^4$.) The Minkowski line element, $ds^2 = \eta_{\mu\nu}\,dx^\mu\,dx^\nu = c^2\,dt^2 - |d\vec{x}|^2$, defines implicitly the fundamental velocity c which need not be equal to the (frequency-dependent) light velocity.

Specifically, the model action is given by

$$S_{\text{photon}} =$$

$$-\tfrac{1}{4} \int_{\mathbb{R}^4} d^4x \ \Big(\eta^{\kappa\mu}\,\eta^{\lambda\nu}\,F_{\mu\nu}(x)\,F_{\kappa\lambda}(x)$$

$$+ \ g(x)\,F_{\kappa\lambda}(x)\,\widetilde{F}^{\kappa\lambda}(x)\Big), \qquad (1)$$

in terms of the Maxwell tensor and

its dual,

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\,, \quad (2a)$$

$$\widetilde{F}^{\kappa\lambda}(x) \equiv \tfrac{1}{2}\,\epsilon^{\kappa\lambda\mu\nu}\,F_{\mu\nu}(x)\,, \qquad (2b)$$

with the completely antisymmetric Levi–Civita symbol $\epsilon^{\kappa\lambda\mu\nu}$ normalized by $\epsilon^{0123} = 1$.

With time-independent "random" couplings $g = g(\vec{x})$ in the model action (1) and a few technical assumptions, the modified dispersion relation is found to be given by:

$$\omega^2 = (1 - A^2\,a_\gamma)\,c^2\,k^2 - A^2\,l_\gamma^2\,c^2\,k^4$$
$$+ \cdots\,, \qquad (3)$$

for wave number $k \equiv |\vec{k}|$.

The constants a_γ and l_γ in (3) are *functionals* of the random couplings $g(\vec{x})$:

$$a_\gamma = \frac{\pi}{18\,A^2}\,C(0)\,, \qquad (4a)$$

$$l_\gamma^2 = \frac{2\pi}{15\,A^2}\,\int_0^\infty \mathrm{d}x\,x\,C(x)\,, \qquad (4b)$$

in terms of the isotropic autocorrelation function

$$C(x) \equiv \widehat{C}(\vec{x})\,, \quad \text{for } x = |\vec{x}|\,, \qquad (5a)$$

with general definition

$$\widehat{C}(\vec{x}) \equiv \lim_{R\to\infty}\ \frac{1}{(4\pi/3)R^3}\int\limits_{|\vec{y}|<R} \mathrm{d}^3\vec{y}$$
$$\times\,g(\vec{y})\,g(\vec{y}+\vec{x})\,. \qquad (5b)$$

For later use, we have also introduced an "amplitude" A of the random couplings $g(\vec{x})$, with perhaps $A \sim \alpha$ from STEP 1. In this way, the quantities a_γ

and l_γ from (4ab) are independent of the overall scale of $g(\vec{x})$.

Note that the calculated dispersion relation (3) does not contain a k^3 term, consistent with general arguments (Lehnert, 2003). The result (3) corresponds to STEP 2 mentioned in the Introduction.

In order to prepare for the first type of experimental limit, we calculate the group velocity $v_g(k) \equiv \mathrm{d}\omega/\mathrm{d}k$ from (3). The relative change between wave numbers k_1 and k_2 is then found to be given by

$$\left.\frac{\Delta c}{c}\right|_{k_1,k_2} \equiv \left|\frac{v_g(k_1) - v_g(k_2)}{v_g(k_1)}\right|$$
$$\sim (3/2)\left|k_1^2 - k_2^2\right|\,A^2\,l_\gamma^2\,, \quad (6)$$

where $\Delta c/c$ is a convenient shorthand notation.

3 Experimental limits

In this section, we obtain bounds from two "gold-plated" events, a TeV gamma-ray flare from an active galactic nucleus (AGN) and an ultra-high-energy cosmic ray (UHECR) from an unknown source.

3.1 TeV γ–ray flare

At the end of Section 2, we have calculated the relative change of the group velocity between two different wave numbers k_i (or photon energies $E_i = \hbar\,\omega_i \sim \hbar\,c\,k_i$). Following the suggestion of Amelino-Camelia et al. (1998), this theoretical result

can be compared with astronomical observations.

In fact, a particular TeV gamma-ray flare of the AGN Markarian 421 provides the following upper bound on $\Delta c/c$ (Biller et al., 1999):

$$\frac{\Delta c}{c}\bigg|^{\text{Mkn 421}}_{\substack{k_1=2.5\times10^{16}\,\text{cm}^{-1} \\ k_2=1.0\times10^{17}\,\text{cm}^{-1}}} < 2.5 \times 10^{-14}\,. \tag{7}$$

Combining the theoretical expression (6) and the astrophysical bound (7) then gives the following "experimental" limit on the photonic length scale (Klinkhamer and Rupp, 2004):

$$l_\gamma < 1.8 \times 10^{-22}\,\text{cm}\ (\alpha/A)\,, \tag{8}$$

with fine-structure constant $\alpha \approx 1/137$ inserted for amplitude A.

See App. A for some details on this experimental limit and for a rough estimate of what might ultimately be achieved with, e.g., the Gamma-ray Large Area Space Telescope (GLAST). For now, bound (8) is our first result for STEP 3 mentioned in the Introduction.

3.2   UHECR

Recall the modified photon dispersion relation (3) and, for definiteness, assume an unmodified proton dispersion relation $E_p^2 = \hbar^2\,c^2\,k^2 + m_p^2\,c^4$.

The absence of Cherenkov-like processes $p \rightarrow p\gamma$ (Coleman and Glashow, 1997) for a proton energy of the order of $E_p \approx 3 \times 10^{11}\,\text{GeV}$ then gives the following experimental limits (Gagnon and Moore, 2004;

Klinkhamer and Rupp, 2005):

$$a_\gamma < \left(6 \times 10^{-19}\right)(\alpha/A)^2\,, \tag{9a}$$

$$l_\gamma < \left(1.0 \times 10^{-34}\,\text{cm}\right)(\alpha/A)\,. \tag{9b}$$

See App. B for the basic physics and astronomy input behind these limits. Bounds (9ab) are the last results for STEP 3 mentioned in the Introduction.

3.3   Three remarks

Having obtained these experimental limits, three remarks are in order. First, bounds (9a) and (9b) arise from soft ($\lesssim$ GeV) and hard (10^{11} GeV) photons, respectively, whereas bound (8) comes from photons with intermediate energies (10^3 GeV).

Second, bound (9b) is twelve orders of magnitude better than (8). In App. A, we show that the potential time-dispersion limit from GLAST would still be far above the Cherenkov limit (9b). This illustrates the power of using ultra-high-energy particles (Coleman and Glashow, 1999), at least for the present purpose.

Third, the Large Hadron Collider (LHC) at CERN will directly probe distances of order 10^{-18} cm, far above the limits from astrophysics. But, then, there is nothing better than a controlled experiment. Clearly, high-energy astrophysics and experimental particle physics are complementary in determining the small-scale structure of spacetime.

4

4 The unbearable smoothness of space

Purely mathematically, define

$$l_\gamma \equiv l_{\text{wormhole}} \left(\frac{l_{\text{wormhole}}}{l_{\text{separation}}} \right)^{3/2}, \quad (10a)$$

$$a_\gamma \equiv \left(\frac{l_{\text{wormhole}}}{l_{\text{separation}}} \right)^3. \quad (10b)$$

For the moment, l_{wormhole} and $l_{\text{separation}}$ are just new symbols.

But, physically, the length l_{wormhole} might correspond to an appropriate characteristic dimension of an *individual* wormhole (e.g., the typical flat-space distance between the centers of the mouths) and the length $l_{\text{separation}}$ to an average separation between *different* wormholes (Klinkhamer and Rupp, 2004). See Fig. 1, but keep in mind that, most likely, the real spacetime cannot be viewed as being embedded in a pseudo-Euclidean space.

From bounds (9a,9b) and definitions (10ab), one gets the exclusion plot of Fig. 2.

It is perhaps not unreasonable to expect some remnant "quantum-gravity" effect with *both* length scales l_{wormhole} and $l_{\text{separation}}$ of the order of the fundamental Planck length (Wheeler, 1968),

$$l_{\text{Planck}} \equiv \hbar c / E_{\text{Planck}} \equiv \sqrt{G \hbar / c^3}$$

$$\approx 1.6 \times 10^{-33} \text{ cm}, \quad (11)$$

with $E_{\text{Planck}} \approx 1.2 \times 10^{19}$ GeV. However, $l_{\text{wormhole}} \sim l_{\text{separation}} \sim 10^{-33}$ cm seems to be ruled out by the limits shown in Fig. 2, provided the amplitude A of the effective ran-
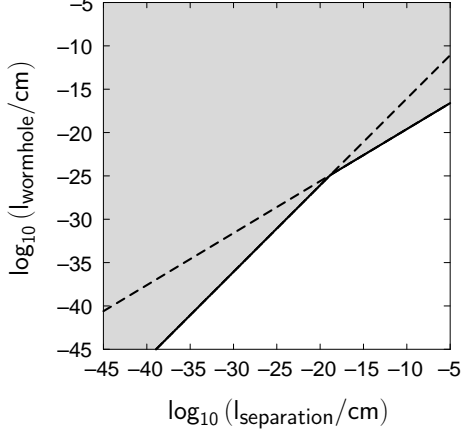

Fig. 2. Excluded region [shaded] for random-coupling amplitude $A = \alpha \approx 1/137$ (see text).

dom coupling $g(x)$ in model (1) is indeed of order $\alpha \approx 1/137$, as suggested by preliminary calculations (Klinkhamer and Rupp, 2004).

The *tentative* conclusion is that a preferred-frame graininess of space with a single length scale l_{Planck} may be hard to reconcile with the current experimental bounds from cosmic-ray physics.

If this conclusion is born out, the question becomes:
why is "empty space" so remarkably smooth?
An alternative form of the same question might read:
why is Lorentz invariance such an accurate symmetry of Nature, perhaps even up to energies of order E_{Planck}?
For further discussion on the question in the second form, see, e.g., Corichi and Sudarsky (2005) and Klinkhamer and Volovik (2005).

Returning to the question in its original form (which may also be related to the dark-energy puzzle), a direct experimental solution would appear to be prohibitively difficult at present, in view of the small numbers

already appearing in (8) and (9b). It may very well be that the only methods available are "indirect," either in the laboratory (e.g., neutrino oscillations) or via astrophysics (e.g., GRBs and UHECRs).

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A Time-dispersion limit

The basic idea (Amelino-Camelia et al., 1998) is to look for time-dispersion effects in a burst-like signal from a very distant astronomical source. The assumption is that the original event was really sharply peaked in time for *all* photon energies simultaneously, excluding the hypothesis of some "cosmic conspiracy."

One particular event turns out to be most useful for our purpose and the observations are shown in Figs. A.1 and A.2. The three peaks of these two figures occur in the same time bin and no time-dispersion is seen. The ratio of the binning interval ($\Delta t \approx 280 \, \text{s}$) over the inferred travel time ($D/c \approx 1.1 \times 10^{16} \, \text{s}$) then gives bound (7) which, in turn, gives bound (8) on the photonic length scale l_γ.

It may be of interest to see what time-dispersion limits can be reached in the future. In order to be specific, we take the most optimistic values for GLAST (McEnery et al., 2004; Bhat et al., 2004). This forthcoming satel-
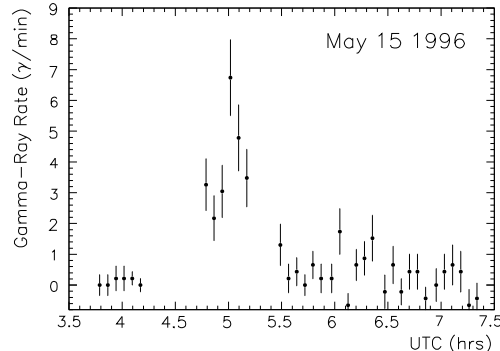

Fig. A.1. TeV γ–ray flare from Markarian 421 observed on May 15, 1996, by the Whipple γ–ray observatory. The rate of excess γ–ray selected events is binned in intervals of 280 seconds. From Biller et al. (1999).

lite experiment has an energy range up to several hundred GeV, may detect gamma-ray bursts (GRBs) up to distances $D \sim 10^{10} \, \text{lyr}$, and has a time resolution of the order of a few microseconds.

From (6), an upper bound $\Delta c/c = c \, \Delta t/D$ for photon energies $E_1 \ll E_2 \equiv E_\text{max}$ then implies the following bound on the photonic length scale:

$$l_\gamma \leq \frac{1}{\sqrt{3/2} \, A} \left(\frac{\hbar c}{E_\text{max}} \right) \left(\frac{c \, \Delta t}{D} \right)^{1/2}$$

$$\approx 1.9 \times 10^{-26} \, \text{cm} \left(\frac{\alpha}{A} \right) \left(\frac{300 \, \text{GeV}}{E_\text{max}} \right)$$

$$\times \left(\frac{\Delta t}{2 \, 10^{-6} \, \text{s}} \right)^{1/2} \left(\frac{3 \, 10^{17} \, \text{s}}{D/c} \right)^{1/2}. \quad \text{(A.1)}$$

This estimate shows that GLAST may indeed be a remarkable probe of small-distance physics (compared to $10^{-18} \, \text{cm}$ from the LHC, for example). Note that the effective length scale probed, l_γ, appears quadratically in the dispersion relation (3), which explains the presence of square roots on the right-hand side of (A.1).
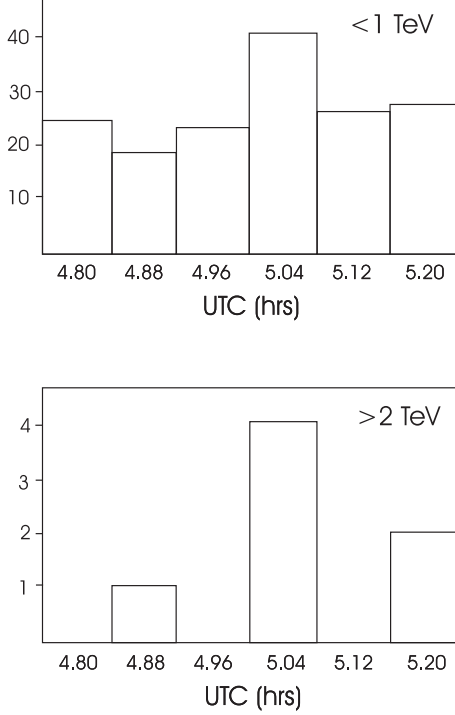
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Fig. A.2. Total number of γ–ray selected events occurring in each 280 second interval near the peak of the 15 May 1996 flare from Markarian 421. The top plot consists of events with γ–ray energies less than 1 TeV, whereas the bottom plot is for energies greater than 2 TeV. From Biller et al. (1999).

B   Cherenkov limits

If the photon dispersion relation is modified, otherwise forbidden photon-radiation processes may become kinematically allowed. We consider, in turn, two different modifications of the photon dispersion relation. Furthermore, we set $\hbar = c = 1$ in this appendix and, for simplicity, assume an unchanged proton dispersion relation, $E_p^2 = k^2 + m_p^2$, with momentum $k \equiv |\vec{k}|$.

The first modification considered corresponds to a possible reduction of the speed of light compared to the maximum attainable speed $c = 1$ of the proton:

$$E_\gamma = (1 - \epsilon)\, k\,, \quad 0 \le \epsilon < 1\,. \quad \text{(B.1)}$$

A charged particle traveling faster than light can now emit "vacuum Cherenkov radiation" (Coleman and Glashow, 1997).

It is, of course, known that hadrons with particularly high energies occur in cosmic rays. As vacuum Cherenkov radiation would slow down charged hadrons traveling through empty space, no hadronic cosmic rays with velocities substantially above the speed of light would ever reach the Earth's atmosphere.

The most energetic cosmic ray reported so far was observed on October 15, 1991, by the Fly's Eye Air Shower Detector in Utah and had $k \approx 3 \times 10^{11}\,\text{GeV}$ (Bird et al., 1995). See Fig. B.1 for the pattern of triggered photomultiplier tubes. Note that several other events with $k \sim 10^{11}\,\text{GeV}$ have been observed by the Akeno Giant Air Shower Array (Takeda et al., 1998).

This particular Fly's Eye event (Bird et al., 1995; Risse et al., 2004) corresponds to a velocity of the assumed primary proton:

$$v \sim 1 - \tfrac{1}{2}\, m_p^2 \,/\, k^2$$
$$\approx 1 - 6 \times 10^{-24}\,, \quad \text{(B.2)}$$

where $m_p \approx 1\,\text{GeV}$ is the proton mass. By direct comparison of (B.2) and (B.1), a rough upper bound on ϵ can be obtained. More precisely, the partonic content of the proton has to be taken into account (Gagnon and Moore, 2004), which leads to a somewhat weaker bound,

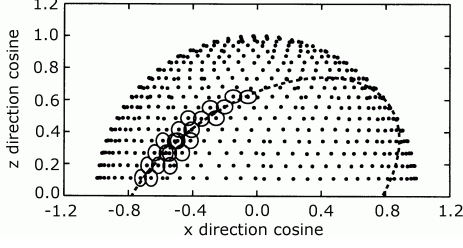$$0 \le \epsilon < 1.6 \times 10^{-23}\,. \quad \text{(B.3)}$$

Fig. B.1. Pointing directions of the 22 photomultiplier tubes which triggered in connection with the Fly's Eye event of October 15, 1991, at 7:34:16 UT. The pointing directions are shown projected into the x–z plane, where the x–axis points east, the y–axis north, and the z–axis upward. The triggered photo-tubes have positive y–components. The dashed line indicates the plane defined by the shower axis and the detector. From Bird et al. (1995).
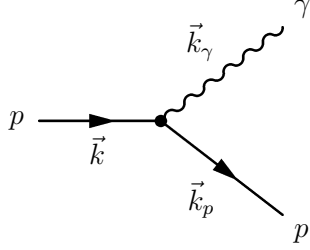

Fig. B.2. Feynman diagram for proton Cherenkov radiation. The heavy dot stands for a momentum-dependent form factor, as the proton p is a composite particle.

Identifying $\epsilon \equiv (1/2)\, A^2\, a_\gamma$, bound (B.3) gives (9a) from the main text.

As the second modification of the photon dispersion relation, consider a contribution to the photon energy which is cubic in the momentum,

$$E_\gamma = k - K_1\, k^3 \,. \tag{B.4}$$

The Cherenkov-like process $p \rightarrow p\,\gamma$ (see Fig. B.2) is then kinematically allowed if the proton energy exceeds a particular threshold energy.

Given the direction of the initial pro-ton momentum, one defines the energy E^{coll} of a decay particle to correspond to the energy of the collinear part of its momentum. Assume, first, that the three-momenta of the two final-state particles are collinear (i.e., $E^{\mathrm{coll}} = E$) and that the photon carries away a fraction x of the initial proton momentum. Due to energy conservation, the difference between final and initial state energy has to vanish, so that

$$\left(E^{\mathrm{coll}}_{\mathrm{final}} - E^{\mathrm{coll}}_{\mathrm{initial}}\right)\Big|^{\substack{\text{collinear}\\ \text{case}}} = 0. \tag{B.5}$$

For a finite opening angle of the final state particles, some energy is used up by the transverse momentum components. One then has the inequality

$$\left(E^{\mathrm{coll}}_{\mathrm{final}} - E^{\mathrm{coll}}_{\mathrm{initial}}\right)\Big|^{\substack{\text{general}\\ \text{case}}} \leq 0. \tag{B.6}$$

Taking $x = 1/2$ and a value $k \approx 3 \times 10^{11}$ GeV, the process shown in Fig. B.2 is allowed if

$$K_1 \geq 5.7\, m_p^2 / k^4$$
$$\approx \left(4 \times 10^{22}\,\mathrm{GeV}\right)^{-2}, \tag{B.7}$$

where, again, the partonic content of the proton has been taken into account (Gagnon and Moore, 2004). Identifying $K_1 \equiv (1/2)\, A^2\, l_\gamma^2$, bound (B.7) gives (9b) from the main text.

Remark, finally, that the bounds of this appendix and Section 3.2 were based on the assumption of having a primary proton p (mass m_p) for the Fly's Eye event considered (see also recent results reported by Abbasi

et al. (2005)). With a primary nucleus N (mass m_N), bounds (9a) and (9b) would increase by approximate factors $(m_N/m_p)^2$ and (m_N/m_p), respectively, as follows by consideration of Eqs. (B.2) and (B.7), neglecting form factors.

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